Common hypercyclic vectors for families of backward shift operators

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Abstract: We provide necessary and sufficient conditions on the existence of common hypercyclic vectors for multiples of the backward shift operator along sparse powers. Our main result strongly generalizes corresponding results which concern the full orbit of the backward shift. Some of our results are valid in a more general context, in the sense that they apply for a wide class of hypercyclic operators.

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1 Introduction

We consider the space ℓ^2 of square summable sequences over the field of complex numbers $\mathbb{C}$ endowed with the topology that is induced by the ℓ^2 norm $\|\cdot\|_2 : \ell^2 \rightarrow \mathbb{R}^+$, where

$$\|x\|_2 := \left(\sum_{j=1}^{+\infty} |x_j|^2 \right)^{1/2} \quad \text{for every} \quad x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell^2.$$

We write $\|\cdot\| := \|\cdot\|_2$ for simplicity. Let B be the unweighted backward shift operator on ℓ^2, that is

$$B(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots), \quad \text{for} \quad (x_1, x_2, \ldots) \in \ell^2.$$

Let $\lambda \in \mathbb{C}$, $|\lambda| > 1$ and consider the set of hypercyclic vectors for λB, that is

$$\mathcal{HC}(\lambda B) := \left\{ x = (x_1, x_2, \ldots) \in \ell^2 \big| \overline{\{(\lambda B)^n(x), \ n = 1, 2, \ldots\}} = \ell^2 \right\}.$$

A comprehensive treatment on hypercyclicity can be found in the books [4], [11]. For the reader's convenience we include the relevant definition. A sequence of continuous

operators (T_n) acting on a Frechet space X is called hypercyclic provided there exists a vector $x \in X$ so that the set $\{T_n(x) : n = 1, 2, \ldots\}$ is dense in X. Such a vector is called hypercyclic for (T_n) and the set of hypercyclic vectors for (T_n) is denoted by $\mathcal{HC}(\{T_n\})$. When T_n comes from the iterates of a single operator we sat that T is hypercyclic and $\mathcal{HC}(T)$ denotes the set of hypercyclic vectors for T, i.e.

$$\mathcal{HC}(T) = \{x \in X : \{T^n x : n = 1, 2, \ldots\} \text{ is dense in } X\}.$$

It is well known that for every $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ the set $\mathcal{HC}(\lambda B)$ is a dense, G_δ subset of $(\ell^2, \|\cdot\|)$ and as the reader may guess Baire's theorem should be involved in the arguments. The following question arises naturally. If we fix an uncountable subset $J \subset \{z \in \mathbb{C} | |z| > 1\}$ is it true that $\bigcap_{\lambda \in J} \mathcal{HC}(\lambda B) \neq \emptyset$? In this direction, Abakumov and Gordon [1] proved that:

$$\bigcap_{|\lambda|>1} \mathcal{HC}(\lambda B) \neq \emptyset,$$

the best possible result one can expect concerning the existence of common hypercyclic vectors for multiples of the backward shift. Later on, Costakis and Sambarino [9] gave a different proof of this result, which, roughly speaking, is based on the so called common hypercyclicity criterion. In this criterion, Baire's category theorem appears. Actually, Costakis and Sambarino showed that $\bigcap_{|\lambda|>1} \mathcal{HC}(\lambda B)$ is a G_δ and dense subset of $(\ell^2, \|\cdot\|)$; hence non-empty. What is interesting here is the uncountable range of λ's, which makes things harder if one wishes to apply Baire's theorem.

One can refine the above problem as follows. Let (k_n) be a fixed subsequence of natural numbers. It is known, and very easy to prove, that the sequence $((\lambda B)^{k_n})$ is also hypercyclic, that is, there exists $x \in \ell^2$ such that the set $\{(\lambda B)^{k_n}(x) : n = 1, 2, \ldots\}$ is dense in ℓ^2. Such a vector is called hypercyclic for $((\lambda B)^{k_n})$ and the set of these vectors is denoted by $\mathcal{HC}(\{(\lambda B)^{k_n}\})$. From the above it should be also clear, or at least expected, that $\mathcal{HC}(\{(\lambda B)^{k_n}\})$ is G_δ and dense subset of $(\ell^2, \|\cdot\|)$.

Now we are ready to ask the following

Question: *Fix a strictly increasing sequence (k_n) of natural numbers. For which uncountable sets $J \subset \{\lambda \in \mathbb{C} : |\lambda| > 1\}$,*

$$\bigcap_{\lambda \in J} \mathcal{HC}(\{(\lambda B)^{k_n}\}) \neq \emptyset ?$$

It turns out that the answer to this question depends heavily on the sequence (k_n). In particular, what matters is how sparse the sequence (k_n) has been chosen. Our main result is the following

Theorem 1.1. *Let (k_n) be a strictly increasing sequence of positive integers.*

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(i) If $\sum_{n=1}^{+\infty} \dfrac{1}{k_n} < +\infty$ then $\bigcap_{\lambda \in I} \mathcal{HC}(\{(\lambda B)^{k_n}\}) = \emptyset$ for every non-degenerate interval in $(1, +\infty)$.

(ii) If $\sum_{n=1}^{+\infty} \dfrac{1}{k_n} = +\infty$ then the set $\bigcap_{\lambda \in (1,+\infty)} \mathcal{HC}(\{(\lambda B)^{k_n}\})$ is residual in ℓ_2, i.e. it contains a G_δ and dense set in ℓ_2; hence $\bigcap_{\lambda \in (1,+\infty)} \mathcal{HC}(\{(\lambda B)^{k_n}\}) \neq \emptyset$.

(iii) If $\sum_{n=1}^{+\infty} \dfrac{1}{k_n} = +\infty$ there exists a G_δ and dense subset $\mathcal{P}$ in $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ with full 2-dimensional Lebesgue measure in $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ such that $\bigcap_{\lambda \in \mathcal{P}} \mathcal{HC}(\{(\lambda B)^{k_n}\})$ is residual in ℓ_2. In particular, $\bigcap_{\lambda \in \mathcal{P}} \mathcal{HC}(\{(\lambda B)^{k_n}\}) \neq \emptyset$.

Unfortunately we are unable to show whether $\mathcal{P}$ in item (iii) of Theorem 1.1 can be replaced by $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$. So, this remains an open problem. On the other hand, both items (i) and (iii) hold in a more general setting (there is nothing special if one choses to work with the backward shift) and this is evident if one follows the relevant proofs, see sections 2 and 4. For instance, the interested readers will have no difficulties in formulating general statements for items (i) and (iii) that involve operators T so that for a given sequence of positive integers (k_n), the sequence$((\lambda T)^{k_n})$ is hypercyclic for every λ lying in some interval or annulus, possibly with infinite length or infinite area. We mention that a kind of similar line of research is pursued in [2], [10], [18], [19], [20], where questions similar to the above one are studied for translation type operators acting on the space of entire functions. Results on the existence of common hypercyclic vectors for uncountable families of operators and, in particular, of backward shift operators can be found in [1], [5], [6], [7], [8], [9], [14], [15], [17]. Our paper is organized as follows. Each one of the following sections 2, 3, 4, is devoted to the proof of items (i), (ii), (iii) of Theorem 1.1 respectively.

The proof of item (i) relies on an estimate which concerns the size (in terms of Lebesgue measure) of the set$\{z \in \mathbb{C}| \ |z^n \cdot w - 1| < \varepsilon\} \cap [m, M]$, for given $w \in \mathbb{C}$, $\varepsilon > 0$, $1 < m < M$. This approach is implicit in [4], [5], [17] and refines an idea of Borichev. The common hypercyclicity criterion due to Costakis and Sambarino cannot be applied in order to conclude item (ii). What we do, is to refine in a sense this criterion in the particular case of backward shift. It seems plausible that our method will possibly work for other operators as well. We mention that there are quite a few, relatively new and powerful, criteria establishing the existence of common hypercyclic vectors for uncountable families of operators, see [4], [5], [17]. However, it is not clear to us whether these criteria can be used in our case. Finally, the proof of item (iii)relies on the following three ingredients: 1) item (ii), 2) a metric result of Weyl which says that, if (k_n) is a given sequence of distinct integers then the sequence $(k_n x)$ is uniformly

distributed mod 1, see Theorem 4.1 in [13], and 3) Cavalieri's principle, see page 149 in [12]. Actually, to prove item (iii) we elaborate on the proof of Proposition 5.2 from [3].

2 A negative result

Fix a subsequence (k_n) of natural numbers such that $\sum_{n=1}^{+\infty} \frac{1}{k_n} < +\infty$. Our goal, in this section, is to show that $\bigcap_{\lambda \in J} \mathcal{H}C(\{(\lambda B)^{k_n}\}) = \emptyset$. First of all we need two helpful lemmas.

Lemma 2.1. *Let $z_0 \in \mathbb{C}$, $N_0 \in \mathbb{N}$, $\varepsilon_0 \in (0,1)$ be three fixed numbers.*
We consider the set

$$G := G(z_0, N_0, \varepsilon_0) = \{z \in \mathbb{C} | \ |z^{N_0} \cdot z_0 - 1| < \varepsilon_0\}.$$

Then G is an open subset of $\mathbb{C}$.

The proof of Lemma 2.1 is straightforward and is omitted. We proceed with the following

Lemma 2.2. *Let $z_0 \in \mathbb{C}$, $N_0 \in \mathbb{N}$, $N_0 > 1$, $\varepsilon_0 \in (0,1)$, $\mu_0 > 1$, $M_0 > \mu_0$ be fixed. Using the notation of the previous lemma, the following estimate holds.*

$$\lambda(G(z_0, N_0, \varepsilon_0) \cap [\mu_0, M_0]) \leq M_0 \cdot \left(\sqrt[N_0]{\frac{1+\varepsilon_0}{1-\varepsilon_0}} - 1 \right).$$

Proof. Suppose that $G_1 := G(z_0, N_0, \varepsilon_0) \cap [\mu_0, M_0] \neq \emptyset$, (the other case is trivial for the proof). Since G_1 is open it contains two different elements a_0 and A_0 where $a_0 < A_0$. By the definition of $G(z_0, N_0, \varepsilon_0)$ it follows that

$$|a_0^{N_0} z_0 - 1| < \varepsilon_0. \tag{1}$$

We have $z_0 \neq 0$. Using (1) and the triangle inequality we get

$$|z_0| > \frac{1-\varepsilon_0}{a_0^{N_0}}. \tag{2}$$

Let $z_0 := t_1 + t_2 i$, where $t_1 = Re(z_0)$ and $t_2 = Im(z_0)$. Then,

$$|a_0^{N_0} z_0 - 1|^2 = |z_0|^2 a_0^{2N_0} + 1 - 2t_1 a_0^{N_0}. \tag{3}$$

By (1) and (3) we conclude that

$$|z_0|^2 (a_0^{N_0})^2 - 2t_1(a_0^{N_0}) + (1 - \varepsilon_0^2) < 0. \tag{4}$$

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Consider the trinomial

$$A(y) := |z_0|^2 y^2 - 2t_1 y + (1 - \varepsilon_0^2).$$

Because $A(a_0^{N_0}) < 0$, by (4) and $|z_0|^2 > 0$ ($z_0 \neq 0$) the trinomial $A(y)$ has positive discriminant $\Delta > 0$ and two roots ρ_1 and ρ_2, where $\rho_1 < \rho_2$ and $A_0^{N_0} < \rho_2$ ($*$) (it follows easily by the above that $\rho_2 > 0$ as well). Hence,

$$\rho_2 \leq \frac{1 + \varepsilon_0}{|z_0|}. \tag{5}$$

By (2), ($*$) and (5) we get

$$A_0^{N_0} < \frac{1 + \varepsilon_0}{1 - \varepsilon_0} \cdot a_0^{N_0} \Rightarrow A_0 - a_0 < M_0 \cdot \left(\sqrt[N_0]{\frac{1 + \varepsilon_0}{1 - \varepsilon_0}} - 1 \right). \tag{6}$$

So, for every $a, A \in G_1$, $a < A$,

$$A - a < M_0 \cdot \left(\sqrt[N_0]{\frac{1 + \varepsilon_0}{1 - \varepsilon_0}} - 1 \right). \tag{6$'$}$$

This gives that

$$\delta(G_1) \leq M_0 \left(\sqrt[N_0]{\frac{1 + \varepsilon_0}{1 - \varepsilon_0}} - 1 \right), \quad \text{where} \quad \delta(G_1) \text{ is the diameter of } G_1 \tag{7}$$

The set G_1 is open in $[\mu_0, M_0]$ and bounded, so

$$G_1 \subseteq [\inf G_1, \sup G_1] \Rightarrow \lambda(G_1) \leq \lambda([\inf G_1, \sup G_1]) = \sup G_1 - \inf G_1 = \delta(G_1). \tag{8}$$

By (7) and (8) we arrive at

$$\lambda(G_1) \leq M_0 \cdot \left(\sqrt[N_0]{\frac{1 + \varepsilon_0}{1 - \varepsilon_0}} - 1 \right)$$

and the proof of Lemma 2.2 is complete. ■

Now we are ready to prove the main theorem of this section.

Theorem 2.3. *Let (k_n) be a subsequence of natural numbers such that $\sum_{n=1}^{+\infty} \frac{1}{k_n} < +\infty$. Then*

$$\bigcap_{\lambda \in (1, +\infty)} \mathcal{H}C(\{(\lambda B)^{k_n}\}) = \emptyset.$$

Proof. We fix two positive numbers μ_0, M_0 such that $1 < \mu_0 < M_0 < +\infty$. It suffices to prove that

$$\bigcap_{\lambda \in [\mu_0, M_0]} \mathcal{H}C(\{(\lambda B)^{k_n}\}) = \emptyset.$$

We set

$$\sigma_0 := \sum_{n=1}^{+\infty} \frac{1}{k_n} < +\infty$$

and fix some positive number $\delta_0 \in \left(0, \dfrac{M_0 - \mu_0}{\sigma_0}\right)$. For instance we may take $\delta_0 := \dfrac{M_0 - \mu_0}{2\sigma_0}$. It is obvious that $\dfrac{\delta_0}{M_0} > 0$, so $e^{\frac{\delta_0}{M_0}} > 1$. Because $\lim_{x \to 0^+} \dfrac{1+x}{1-x} = 1$, there exists some positive number $\varepsilon_0 \in (0, 1)$ such that

$$\frac{1+\varepsilon_0}{1-\varepsilon_0} < e^{\frac{\delta_0}{M_0}}. \tag{1}$$

We now fix some positive number $\varepsilon_0 \in (0, 1)$ such that (1) holds.

There exists some natural number $N_0 \in \mathbb{N}$ such that

$$\left(\frac{\delta_0}{M_0 N} + 1\right)^N > \frac{1+\varepsilon_0}{1-\varepsilon_0} \tag{2}$$

for every $N \in \mathbb{N}$, $N \geq N_0$, by (1). To arrive at a contradiction, suppose that $\bigcap_{\lambda \in [\mu_0, M_0]} \mathcal{HC}(\{(\lambda B)^{k_n}\}) \neq \emptyset$. We fix some

$$x_0 := (x_1, x_2, \ldots) \in \bigcap_{\lambda \in [\mu_0, M_0]} \mathcal{HC}(\{(\lambda B)^{k_n}\})$$

and let $e_1 := (1, 0, 0, \ldots) \in \ell^2$. We fix some $\lambda_0 \in [\mu_0, M_0]$. Since

$$x_0 \in \bigcap_{\lambda \in [\mu_0, M_0]} \mathcal{HC}(\{(\lambda B)^{k_n}\}),$$

there exists a subsequence (μ_n) of (k_n) such that $(\lambda_0 B)^{\mu_n}(x_0) \to e_1$. For $\varepsilon_1 := \varepsilon_0/2$, there exists some natural number $n_0 \geq N_0$ such that

$$\|(\lambda_0 B)^{\mu_n}(x_0) - e_1\| < \varepsilon_1 \quad \text{for every} \quad n \geq n_0. \tag{3}$$

By (3) we get

$$|\lambda_0^{\mu_n} x_{\mu_n + 1} - 1| < \varepsilon_1 \quad \text{for every} \quad n \in \mathbb{N}, \quad n \geq n_0. \tag{4}$$

Though λ_0 is fixed, we apply the above for every $\lambda \in [\mu_0, M_0]$. Hence, (4) implies that for every $\lambda \in [\mu_0, M_0]$ there exists some natural number $v \geq N_0$ such that

$$|\lambda^{k_v} x_{k_v + 1} - 1| < \varepsilon_1. \tag{5}$$

Setting

$$L := \left\{\lambda \in [\mu_0, M_0] / \exists\, v \geq N_0 : |\lambda^{k_v} x_{k_v + 1} - 1| < \varepsilon_1\right\},$$

we get $L = [\mu_0, M_0]$, by (5). Consider the set

$$\mathcal{N}_1 := \big\{ v \in \mathbb{N} / v \geq N_0 \text{ and } \exists\, \lambda \in [\mu_0, M_0] \text{ such that } |\lambda^{k_v} x_{k_v+1} - 1| < \varepsilon_1 \big\}.$$

Then, $\mathcal{N}_1 \neq \emptyset$ by (5).

Let $v \in \mathcal{N}_1$ and define the set

$$G_v := \big\{ \lambda \in [\mu_0, M_0] \,\big|\, |\lambda^{k_v} x_{k_v+1} - 1| < \varepsilon_1 \big\}.$$

It is obvious that $G_v \neq \emptyset$ for every $v \in \mathcal{N}_1$. By Lemma 2.1, G_v is open in $[\mu_0, M_0]$ for every $v \in \mathcal{N}_1$ and

$$[\mu_0, M_0] = \bigcup_{v \in \mathcal{N}_1} G_v. \tag{6}$$

By the properties of Lebesgue measure, Lemma 2.2 and (6) we have

$$M_0 - \mu_0 = \lambda([\mu_0, M_0]) = \lambda\bigg(\bigcup_{v \in \mathcal{N}_1} G_v \bigg) \leq \sum_{v \in \mathcal{N}_1} \lambda(G_v)$$

$$\leq \sum_{v \in \mathcal{N}_1} M_0 \bigg(\sqrt[k_v]{\frac{1 + \varepsilon_1}{1 - \varepsilon_1}} - 1 \bigg)$$

$$\leq M_0 \cdot \sum_{v=N_0}^{+\infty} \bigg(\sqrt[k_v]{\frac{1 + \varepsilon_1}{1 - \varepsilon_1}} - 1 \bigg). \tag{7}$$

Observe that $\dfrac{1 + \varepsilon_0}{1 - \varepsilon_0} > \dfrac{1 + \varepsilon_1}{1 - \varepsilon_1}$, since

$$\varepsilon_1 \in (0, \varepsilon_0). \tag{8}$$

By (2) and (8) we get

$$\sqrt[k_v]{\frac{1 + \varepsilon_1}{1 - \varepsilon_1}} - 1 < \frac{\delta_0}{M_0 k_v} \quad \text{for every} \ \ v \geq N_0 \tag{9}$$

and (7), (9) imply that

$$M_0 \cdot \sum_{v=N_0}^{+\infty} \bigg(\sqrt[k_v]{\frac{1 + \varepsilon_1}{1 - \varepsilon_1}} - 1 \bigg) < \delta_0 \cdot \sum_{v=N_0}^{+\infty} \frac{1}{k_v} < M_0 - \mu_0 \tag{10}$$

(by the definition of δ_0). Obviously, the inequalities (7) and (10) are in contradiction and this completes the proof of Theorem 2.3. ∎

3 The positive case in the half line $(1, +\infty)$

Throughout this section we fix a subsequence (k_n) of natural numbers such that $\sum_{n=1}^{+\infty} \frac{1}{k_n} = +\infty$. We shall prove the following

Theorem 3.1. *The set $\bigcap_{\lambda \in (1,+\infty)} \mathcal{HC}(\{(\lambda B)^{k_n}\})$ is a residual subset of $(\ell^2, \|\cdot\|)$.*

In order to prove Theorem 3.1 we assign some notations and terminology. Let $D := \big\{ x = (x_1, x_2, \ldots) \in \ell^2 \big| \{n \in \mathbb{N} | x_n \neq 0\}$ is a finite subset of $\mathbb{N}$ and $x_n \in \mathbb{Q} + i\mathbb{Q}$ for every $n = 1, 2, \ldots \big\}$, where $\mathbb{Q}$ is the set of rational numbers, The set D is countable and dense in $(\ell^2, \|\cdot\|_2)$. We set $\overline{0} := (0, 0, \ldots) \in \ell^2$ and $D^* := D \smallsetminus \{\overline{0}\}$. Let $\Psi := \{y_1, y_2, y_3, \ldots, y_n, \ldots\}$ be an enumeration of D^*. We fix a strictly decreasing sequence of positive numbers (a_n) such that $a_n \to 1$ (for example $a_n := 1 + \frac{1}{n}$, $n = 1, 2, \ldots$) and we also fix a strictly increasing sequence (β_n) of positive numbers such that $\beta_n \to +\infty$ and $a_1 < \beta_1$ (for example $\beta_n = n + 2$, $n = 1, 2, \ldots$). Then we set $\Delta_n := [a_n, \beta_n]$, $n = 1, 2, \ldots$. Of course, the sequence of compact sets Δ_n, $n = 1, 2, \ldots$ forms an exhausting family of $(1, +\infty)$, i.e. $(1, +\infty) = \bigcup_{n=1}^{+\infty} \Delta_n$.

For every $n, j, s, m \in \mathbb{N}$ let us define

$$E_{\Delta_n}(j, s, m) := \{ x = (x_1, x_2, \ldots) \in \ell^2 |$$

for every $\lambda \in \Delta_n$ there exists some $v \in \mathbb{N}$, $v \leq m$ such that

$$\|(\lambda B)^{k_v}(x) - y_j\| < \frac{1}{s} \Big\}.$$

We finally set

$$G := \bigcap_{n=1}^{+\infty} \bigcap_{j=1}^{+\infty} \bigcap_{s=1}^{+\infty} \bigcup_{m=1}^{+\infty} E_{\Delta_n}(j, s, m).$$

Lemma 3.2. *For every $n, j, s, m \in \mathbb{N}$ the set $E_{\Delta_n}(j, s, m)$ is open in $(\ell^2, \|\cdot\|)$.*

Proof. The proof is easy and we leave it to the reader.

Lemma 3.3. *We have*

$$G \subseteq \bigcap_{\lambda \in (1,+\infty)} \mathcal{HC}(\{(\lambda B)^{k_n}\})$$

Proof. The proof is trivial. ■

We proceed with two more lemmas that lie in the heart of the argument.

Lemma 3.4. *Let a_0, b_0 be two positive real numbers such that $1 < a_0 < b_0 < +\infty$. Let (k_n) be a subsequence of natural numbers such that $\sum_{n=1}^{+\infty} \frac{1}{k_n} = +\infty$. Then for every positive number $\varepsilon > 0$ there exists a finite number of terms $k_{n_0}, k_{n_0+1},$*

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$\ldots, k_{n_0+i_0}$ of (k_n) for some natural numbers n_0, i_0 and positive numbers $\beta_1, \beta_2, \ldots, \beta_{i_0+1}$ such that: for every $\lambda \in [a_0, b_0]$, there exists some $j \in \{0, 1, \ldots, i_0\}$ such that

$$|\lambda^{k_{n_0+j}} \beta_{j+1} - 1| < \varepsilon.$$

Proof. We fix some positive number $\varepsilon_0 \in (0, 1)$. After we fix some natural number n_0 such that:

$$k_{n_0} > \frac{\log(1 + \varepsilon_0)}{\log\left(\frac{b_0}{a_0}\right)}. \tag{1}$$

Of course $\sum_{j=0}^{+\infty} \frac{1}{k_{n_0+j}} = +\infty$. So by (1) there exists the unique natural number $i_0 \in \{0, 1, 2, \ldots\}$ such that

$$(1 + \varepsilon_0)^{\sum_{j=0}^{i_0} \frac{1}{k_{n_0+j}}} \leq \frac{b_0}{a_0} \quad \text{and} \tag{2}$$

$$(1 + \varepsilon_0)^{\sum_{j=0}^{i_0+1} \frac{1}{k_{n_0+j}}} > \frac{b_0}{a_0}. \tag{3}$$

We set $\beta_1 := \frac{1}{a^{k_{n_0}}}$. Then for every $\lambda \in [a_0, b_0]$ such that $\lambda < a_0 \cdot \sqrt[k_{n_0}]{1 + \varepsilon_0}$, we get $|\lambda^{k_{n_0}} \cdot \beta_1 - 1| < \varepsilon_0$.

After we set $a_1 := a_0 \cdot \sqrt[k_{n_0}]{1 + \varepsilon_0}$ and $\beta_2 := \frac{1}{a_1^{k_{n_0+1}}}$.

Then, for every $\lambda \in [a_1, b_0]$ with $\lambda < a_1 \cdot \sqrt[k_{n_0+1}]{1 + \varepsilon_0}$ we have $|\lambda^{k_{n_0+1}} \beta_2 - 1| < \varepsilon_0$. We continue inductively.

We suppose that we have defined the number $a_i = a_0(1 + \varepsilon_0)^{\sum_{j=0}^{i-1} \frac{1}{k_{n_0+j}}}$ for some $i \in \{1, 2, \ldots, i_0 - 1\}$.

After we define $\beta_{i+1} := 1/a_i^{k_{n_0+i}}$ and for every $\lambda \in [a_i, b_0]$ with $\lambda < a_i \cdot \sqrt[k_{n_0+i}]{1 + \varepsilon_0}$ we get $|\lambda^{k_{n_0+i}} \cdot \beta_{i+1} - 1| < \varepsilon_0$.

Because $a_0 < a_1 < a_2 < \cdots < a_{i_0} \leq b_0$ and a_{i_0} is the maximum number in $(a_0, b_0]$ that we can obtain with the above procedure, after a finite number of steps we exclude the interval $[a_0, b_0]$ and we get the conclusion of this lemma with the following data:

We have

$$a_i = a_0 \cdot \prod_{j=0}^{i-1} (1 + \varepsilon_0)^{\frac{1}{k_{n_0+j}}} \quad \text{for every} \quad i = 1, 2, \ldots, i_0$$

$$\beta_{i+1} = 1/a_i^{k_{n_0+i}} \quad \text{for every} \quad i = 0, 1, \ldots, i_0.$$

It follows that for every $\lambda \in [a_i, b_0]$ where $\lambda < a_i \cdot \sqrt[k_{n_0+i}]{1 + \varepsilon_0}$ we have:

$$|\lambda^{k_{n_0+i}} \beta_{i+1} - 1| < \varepsilon_0 \quad \text{for every} \quad i = 0, 1, \ldots, i_0.$$

With the above data the proof of this lemma is completed. ∎

Lemma 3.5. *Let (k_n) be a subsequence of natural numbers such that $\sum\limits_{n=1}^{+\infty} \dfrac{1}{k_n} = +\infty$.*

Then for every positive number $M > 0$ there exists a subsequence (μ_n) of (k_n) such that:

(i) $\mu_{n+1} - \mu_n > M$ *for every* $n = 1, 2, \ldots$ *and*

(ii) $\sum\limits_{n=1}^{+\infty} \dfrac{1}{\mu_n} = +\infty.$

Proof. We fix some positive number $M_0 > 1$.

Let $N_0 := [M_0] + 1 \geq 2$ (where $[x]$ is the integer part of $x \in \mathbb{R}$).

We consider the subsequences $(\mu_\rho^j)_{\rho=1,2,\ldots} = (k_{\rho N_0 + j})_{\rho=1,2,\ldots}$ of (k_n) for every $j = 0, 1, \ldots, N_0 - 1$, that is $\mu_\rho^j := k_{\rho N_0 + j}$, $\rho = 1, 2, \ldots$ for $j = 0, 1, \ldots, N_0 - 1$.

We fix some $j_0 \in \{0, 1, \ldots, N_0 - 1\}$. We consider the subsequence $(\mu_\rho^{j_0})_{\rho=1,2,\ldots}$ of (k_n).

Claim 1.

For every $v_1, v_2 \in \mathbb{N}$, $v_1 < v_2$ we have: $k_{v_2} - k_{v_1} \geq v_2 - v_1$.

Proof of the Claim 1.

Because (k_n) is a subsequence of natural numbers we have:

$$k_{v_1+1} - k_{v_1} \geq 1, \ k_{v_1+2} - k_{v_1+1} \geq 1, \ldots,$$

$$k_{v_1} + (v_2 - v_1) - k_{v_1} + (v_2 - v_1) - 1 \geq 1$$

Adding the above inequalities we take the conclusion of Claim 1.

We apply Claim 1 for the terms of subsequence $(\mu_\rho^{j_0})_{\rho=1,2,\ldots}$ and we have:

$$\mu_{\rho+1}^{j_0} - \mu_\rho^{j_0} > M_0. \tag{1}$$

By (1) we have that each one from the subsequences $(\mu_\rho^j)_{\rho=1,2,\ldots}$ for $j = 0, 1, \ldots, N_0 - 1$ of (k_n) satisfies (1). It is obvious, by the definition of the sequences $(\mu_\rho^j)_{\rho=1,2,\ldots}$, for $j = 0, 1, \ldots, N_0 - 1$ that these do not have common terms by pairs and every term of the sequence (k_v) belongs in a unique subsequence $(\mu_\rho^j)_{\rho=1,2,\ldots}$ for some $j \in \{0, 1, \ldots, N_0 - 1\}$.

This gives that

$$\sum_{n=1}^{+\infty} \frac{1}{k_n} = \sum_{j=0}^{N_0-1} \sum_{\rho=1}^{+\infty} \frac{1}{\mu_\rho^j}. \tag{2}$$

Because $\sum\limits_{n=1}^{+\infty} \dfrac{1}{k_n} = +\infty$, the relation (2) gives us that there exists one $j \in \{0, 1, \ldots, N_0 - 1\}$ at least such that $\sum\limits_{\rho=1}^{+\infty} \dfrac{1}{\mu_\rho^j} = +\infty$, where the subsequence (μ_ρ^j) of (k_n) satisfies the properties (i) and (ii) obviously and this completes the proof of this lemma. ∎

After the above preparation we are ready now to prove the following lemma that is the basic result that gives us Theorem 3.1.

Lemma 3.6. *For every $n, j, s \in \mathbb{N}$ the set $\bigcup\limits_{m=1}^{+\infty} E_{\Delta_n}(j, s, m)$ is dense in $(\ell^2, \|\cdot\|)$.*

Proof. We fix $n_0, j_0, s_0 \in \mathbb{N}$ and we will show that the set $\bigcup\limits_{m=1}^{+\infty} E_{\Delta_{n_0}}(j_0, s_0, m)$ is dense in $(\ell^2, \|\cdot\|)$.

We set $E := \bigcup\limits_{m=1}^{+\infty} E_{\Delta_{n_0}}(j_0, s_0, m)$ for simplicity.

Let $y_{j_0} := (q_1, q_2, \ldots, q_{v_0}, 0, 0, \ldots)$ where $q_{v_0} \neq 0$, $y_{j_0}(v) = q_v$, for every $v = 1, 2, \ldots$, $y_{j_0}(v) = 0$ for every $v \geq v_0 + 1$, and $q_j \in \mathbb{Q} + i\mathbb{Q}$ for every $j = 1, 2, \ldots, v_0$, for some fixed $v_0 \in \mathbb{N}$.

We fix $\varepsilon_0 > 0$ and $c_0 = (c_1, c_2, \ldots, c_{v_1}, 0, 0, \ldots) \in D$ where $c_{v_1} \neq 0$, $v_1 \in \mathbb{N}$, fixed, $c_0(v) = c_v$ for every $v = 1, 2, \ldots$, $c_0(v) = 0$ for every $v \geq v_1 + 1$, $c_j \in \mathbb{Q} + i\mathbb{Q}$ for every $j = 1, 2, \ldots, v_1$. We consider the ball $S_{\ell^2}(c_0, \varepsilon_0) := \{x \in \ell^2 \,|\, \|x - c_0\| < \varepsilon_0\}$.

We will show that

$$E \cap S_{\ell^2}(c_0, \varepsilon_0) \neq \emptyset. \tag{$*$}$$

In order to show the relation $(*)$ it suffices to show that there exists some $x_0 = (x_1, x_2, \ldots, x_n, \ldots) \in \ell^2$ and $m_0 \in \mathbb{N}$ such that

(i) $\|x_0 - c_0\| < \varepsilon_0$ and

(ii) for every $\lambda \in \Delta_{n_0}$ there exists some $v \in \mathbb{N}$, $v \leq m_0$ such that

$$\|(\lambda B)^{k_v}(x_0) - y_{j_0}\| < \frac{1}{s_0}. \tag{1}$$

We will succeed (i) and (ii) above as follows:

From the data of the problem we define a finite number of complex numbers x_j, $j = 1, 2, \ldots, \ell_0$ for some fixed $\ell_0 \in \mathbb{N}$.

Afterwards, we define the sequence $x_0 := (x_1, x_2, \ldots, x_{\ell_0}, 0, 0)$ where $x_0(j) = x_j$ for every $j = 1, 2, \ldots, \ell_0$ and $x_0(j) = 0$ for every $j \geq \ell_0 + 1$. So we have $x_0 \in \ell^2$.

We define also a natural number m_0.

Finally, we show that x_0 and m_0 satisfy properties (i) and (ii) of (1) as above.

Without loss of generality let $\Delta_{n_0} := [a_0, b_0]$, where $1 < a_0 < b_0 < +\infty$. We set

$$M_1 := \max\{|q_j|, \ j = 1, 2, \ldots, v_0\} > 0.$$

We also set

$$M_2 := \frac{1}{2 \log a_0} \cdot \log\left(\frac{2s_0^2 \cdot M_1^2 \cdot v_0}{1 - \dfrac{1}{a_0}}\right).$$

By Lemma 3.5 we choose a subsequence (μ_n) of (k_n) such that the following two properties hold:

(i) $\mu_{n+1} - \mu_n > \max\{M_2, v_0\}$ for every $n = 1, 2, \dots$ and

(ii) $\sum\limits_{n=1}^{+\infty} \dfrac{1}{\mu_n} = +\infty$.

We set $\varepsilon_1 := \dfrac{1}{\sqrt{2v_0} \cdot s_0 \cdot M_1}$.

Now we can choose some fixed natural number $v_2 \in \mathbb{N}$ such that the following three inequalities hold:

a) $\mu_{v_2} > v_1 + 1$

b) $\mu_{v_2} > \dfrac{\log(1 + \varepsilon_1)}{\log\left(\dfrac{b_0}{a_0}\right)}$

c) $\mu_{v_2} > \dfrac{1}{2 \log a_0} \cdot \log\left(\dfrac{v_0 \cdot M_1^2}{\varepsilon_0^2 \cdot \left(1 - \dfrac{1}{a_0}\right)} \right)$.

Because $\sum\limits_{n=1}^{+\infty} \dfrac{1}{\mu_n} = +\infty$ we have $\sum\limits_{j=0}^{+\infty} \dfrac{1}{\mu_{v_2+j}} = +\infty$.

Let i_0 be the unique natural number $i_0 \in \{0, 1, 2, \dots\}$ such that

$$(1 + \varepsilon_1)^{\sum\limits_{j=0}^{i_0} \frac{1}{\mu_{v_2+j}}} \leq \frac{b_0}{a_0} \quad \text{and} \quad (1 + \varepsilon_1)^{\sum\limits_{j=0}^{i_0+1} \frac{1}{\mu_{v_2+j}}} > \frac{b_0}{a_0} \tag{2}$$

We set $m_1 := v_2 + i_0$. The natural number m_1 is the natural number in order (1) holds. That is let m_0 be the unique natural number such that: $k_{m_0} := \mu_{m_1}$. Then m_0 is the natural number we search in (1).

Now, we are ready to define the vector $x_0 = (x_1, x_2, \dots) \in \ell^2$ straightforward with full details.

We define $x_j = c_j$ for every $j = 1, 2, \dots, v_1$. We define $x_j = 0$ for every $j \in \mathbb{N}$ such that: $v_1 + 1 \leq j \leq \mu_{v_2}$.

Because $\sum\limits_{n=1}^{+\infty} \dfrac{1}{\mu_n} = +\infty$, we can apply Lemma 3.4 for the sequence (μ_n). By the relations (b) and (2) above that the numbers v_2 and ε_1, i_0 satisfy and using Lemma 3.4 for the sequence (μ_n) we take that there exists a finite number of positive numbers $\beta_1, \beta_2, \dots, \beta_{i_0+1}$ that are defined completely in Lemma 3.4 from our data such that for every $\lambda \in [a_0, b_0]$ there exists unique $j \in \{0, 1, \dots, i_0\}$ such that

$$|\lambda^{\mu_{v_2}+j} \beta_{j+1} - 1| < \varepsilon_1. \tag{3}$$

We remark by the previous Lemma 3.4 that for every $\lambda \in [a_0, b_0]$ there exists unique a_i, $i \in \{0, 1, \dots, i_0\}$ such that $a_i \leq \lambda \leq a_{i+1}$ if $i \leq i_0 - 1$ or $a_{i_0} \leq \lambda \leq b_0$, where the numbers a_i, $i = 0, 1, \dots, i_0$ are defined completely by Lemma 3.4.

So, we have defined completely the positive numbers β_{j+1}, for $j = 0, 1, \dots, i_0$.

Now, we define $x_{\mu_{v_2}+i+j} = \beta_{i+1} q_j$ for every $i = 0, 1, \ldots, i_0$ and for every $j = 1, 2, \ldots, v_0$.

The previous terms $x_{\mu_{v_2}+i+j}$ for $i = 0, 1, \ldots, i_0$, $j = 1, 2, \ldots, v_0$ are defined well because $\mu_{n+1} - \mu_n > v_0$ for every $n = 1, 2, \ldots$ by the definition of the sequence (μ_n).

Finally, we define that $x_j = 0$ for every $j \in \mathbb{N}$, $j > \mu_{v_2}$ for which there exists not $i \in \{0, 1, \ldots, i_0\}$ and $j_1 \in \{1, 2, \ldots, v_0\}$ such that $j = \mu_{v_2+i} + j_1$.

By the previous procedure we have defined completely the vector

$x_0 = (x_1, x_2, \ldots, x_n, \ldots)$ where $\{n \in \mathbb{N} \,|\, x_n \neq 0\}$ is finite and thus $x_0 \in \ell^2$ obviously.

Now, we show that the vector x_0 satisfies relation (1).

Firstly we prove that $x_0 \in S_{\ell^2}(c_0, \varepsilon_0)$.

By the definition of the vector x_0 we get:

$$
\begin{aligned}
\|x_0 - c_0\|^2 &:= \sum_{j=1}^{+\infty} |x_j - c_j|^2 = \sum_{j=v_1+1}^{+\infty} |x_j|^2 \\
&= \sum_{j=1}^{v_0} \sum_{i=0}^{i_0} |x_{\mu_{v_2}+i+j}|^2 \\
&= \sum_{j=1}^{v_0} \sum_{i=0}^{i_0} |\beta_{i+1} q_j|^2 = \sum_{j=1}^{v_0} |q_j|^2 \sum_{i=0}^{i_0} |\beta_{i+1}|^2 \\
&\leq v_0 M_1^2 \cdot \sum_{i=0}^{i_0} |\beta_{i+1}|^2 = v_0 M_1^2 \cdot \sum_{i=0}^{i_0} \frac{1}{a_i^{2\mu_{v_2}+i}} \\
&\leq v_0 M_1^2 \cdot \sum_{i=0}^{i_0} \frac{1}{a_0^{2\mu_{v_2}+i}} < v_0 M_1^2 \cdot \sum_{v=2\mu_{v_2}}^{+\infty} \frac{1}{a_0^v} \\
&= v_0 M_1^2 \frac{1}{a_0^{2\mu_{v_2}}} \cdot \frac{1}{1 - \dfrac{1}{a_0}} < \varepsilon_0^2
\end{aligned}
\tag{4}
$$

by the inequality (c) above for μ_{v_2}.

Inequality (4) gives that $x_0 \in S_{\ell^2}(c_0, \varepsilon_0)$, so property (i) of (1) holds.

We show now that property (ii) also holds.

We fix some $\lambda \in [a_0, b_0]$. Then there exists unique $\rho_0 \in \{1, 2, \ldots, i_0 - 1\}$ such that $a_{\rho_0} \leq \lambda < a_{\rho_0} \cdot (1 + \varepsilon_0)^{\frac{1}{\mu_{v_2}+\rho_0}}$ or $a_{i_0} \leq \lambda \leq b_0$.

We show that

$$
\|(\lambda B)^{\mu_{v_2}+\rho_0}(x_0) - y_{j_0}\| < \frac{1}{s_0}.
$$

We have:

$$\|(\lambda B)^{\mu_{v_2}+\rho_0}(x_0) - y_{j_0}\|^2 = \sum_{j=1}^{v_0} |\lambda^{\mu_{v_0}+\rho_0} x_{\mu_{v_2}+\rho_0+j} - q_j|^2$$

$$+ \sum_{j=v_0+1}^{+\infty} |\lambda^{\mu_{v_2}+\rho_0} x_{\mu_{v_2}+\rho_0+j}|^2. \qquad (5)$$

By definition we have for $j = 1, 2, \ldots, v_0$ $x_{\mu_{v_2}+\rho_0+j} = \beta_{\rho_0+1} q_j$. So, for $j = 1, 2, \ldots, v_0$ and $\lambda \in [a_{\rho_0}, a_{\rho_0+1}]$, where $a_{\rho_0+1} = a_{\rho_0} \cdot (1+\varepsilon_1)^{\frac{1}{\mu_{v_2}+\rho_0}}$ or $\lambda \in [a_{\rho_0}, b_0]$ if $\rho_0 = i_0$ we have

$$|\lambda^{\mu_{v_2}+\rho_0} x_{\mu_{v_2}+\rho_0+j} - q_j|^2 = |\lambda^{\mu_{v_2}+\rho_0} \beta_{\rho_0+1} q_j - q_j|^2$$

$$= |\lambda^{\mu_{v_2}+\rho_0} \beta_{\rho_0+1} - 1|^2 |q_j|^2$$

$$\leq |\lambda^{\mu_{v_2}+\rho_0} \beta_{\rho_0+1} - 1|^2 \cdot M_1^2 < \varepsilon_1^2 M_1^2 = \frac{1}{2v_0 s_0^2}.$$

So we have:

$$\sum_{j=1}^{v_0} |\lambda^{\mu_{v_2}+\rho_0} x_{\mu_{v_2}+\rho_0+j} - q_j|^2 < \frac{1}{2s_0^2}. \qquad (6)$$

If $\rho_0 = i_0$ the second member of (5) is 0 and the conclusion holds by (6).

So for the sequel we suppose that $\rho_0 \leq i_0 - 1$. In this case we get

$$\sum_{j=v_0+1}^{+\infty} |\lambda^{\mu_{v_2}+\rho_0} x_{\mu_{v_2}+\rho_0+j}|^2 = \lambda^{2\mu_{v_2}+\rho_0} \sum_{j=1}^{v_0} \sum_{i=1}^{i_0-\rho_0} |x_{\mu_{v_2}+\rho_0+i+j}|^2$$

$$= \lambda^{2\mu_{v_2}+\rho_0} \sum_{j=1}^{v_0} |q_j|^2 \sum_{i=1}^{i_0-\rho_0} |\beta_{\rho_0+i+1}|^2$$

$$\leq \lambda^{2\mu_{v_2}+\rho_0} v_0 M_1^2 \sum_{i=1}^{i_0-\rho_0} |\beta_{\rho_0+i+1}|^2$$

$$= \lambda^{2\mu_{v_2}+\rho_0} v_0 M_1^2 \sum_{i=1}^{i_0-\rho_0} \frac{1}{\left(a_{\rho_0+i}^{\mu_{v_2}+\rho_0+i}\right)^2}$$

$$\leq \lambda^{2\mu_{v_2}+\rho_0} v_0 M_1^2 \sum_{i=1}^{i_0-\rho_0} \frac{1}{a_{\rho_0+1}^{2\mu_{v_2}+\rho_0+i}}$$

$$< \lambda^{2\mu_{v_2}+\rho_0} v_0 M_1^2 \cdot \sum_{v=2\mu_{v_2}+\rho_0+1}^{+\infty} \frac{1}{a_{\rho_0+1}^{v}}$$

$$= \lambda^{2\mu_{v_2}+\rho_0} v_0 M_1^2 \frac{1}{a_{\rho_0+1}^{2\mu_{v_2}+\rho_0+1}} \cdot \frac{1}{1 - \frac{1}{a_{\rho_0+1}}}$$

$$< a_{\rho+1}^{2\mu_{v_2}+\rho} v_0 M_1^2 \frac{1}{a_{\rho_0+1}^{2\mu_{v_2}+\rho_0+1}} \cdot \frac{1}{1-\dfrac{1}{a_0}}$$

$$= \frac{v_0 M_1^2}{1-\dfrac{1}{a_0}} \cdot \frac{1}{a_{\rho_0+1}^{2(\mu_{v_2}+\rho_0+1-\mu_{v_2}+\rho_0)}}$$

$$< \frac{v_0 M_1^2}{1-\dfrac{1}{a_0}} \cdot \frac{1}{a_0^{2(\mu_{v_2}+\rho_0+1-\mu_{v_2}+\rho_0)}} < \frac{1}{2s_0^2} \tag{7}$$

because $\mu_{v+1} - \mu_v > M_2$ for every $v \geq v_2$ by the hypothesis (i) for the sequence (μ_n).

By (5), (6) and (7) we get that $\|(\lambda B)^{\mu_{v_2}+\rho_0}(x_0) - y_{j_0}\| < \dfrac{1}{s_0}$ for the arbitrary $\lambda \in [a_0, b_0]$. This completes property (ii) of (1) and the proof of this Lemma 3.6 is completed. ∎

Now by Lemmas 3.2, 3.3, 3.6 and the facts that the space $(\ell^2, \|\cdot\|)$ is a complete metric space and Baire's category Theorem the proof of Theorem 3.1 is completed.

4   A result in measure and category

In this section we prove item (iii) of Theorem 1.1. Actually, we shall prove the following, more general, result. As we already mentioned in the Introduction, its proof elaborates on the proof of Proposition 5.2 from [3].

Theorem 4.1. *Let (k_n) be a strictly increasing sequence of positive integers. Let T be a bounded linear operator acting on a (complex) Banach space X such that $((\lambda T)^{k_n})$ is hypercyclic for every $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ and assume in addition that*

$$\bigcap_{\lambda \in (1, +\infty)} \mathcal{HC}(\{(\lambda T)^{k_n}\}) \neq \emptyset.$$

Then, there exists a G_δ and dense subset $\mathcal{P}$ in $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ with full 2-dimensional Lebesgue measure in $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ such that $\bigcap_{\lambda \in \mathcal{P}} \mathcal{HC}(\{(\lambda T)^{k_n}\})$ is residual in X. In particular, $\bigcap_{\lambda \in \mathcal{P}} \mathcal{HC}(\{(\lambda T)^{k_n}\}) \neq \emptyset$.

Proof. By performing a change of variables it suffices to prove the following:

Claim. *Fix $x \in \bigcap_{\lambda \in (1, +\infty)} \mathcal{HC}(\{(\lambda T)^{k_n}\})$. Then there exists a G_δ and dense subset A of $(1, +\infty) \times \mathbb{R}$ with full (2-dimensional) Lebesgue measure such that the set $\{((rT)^{k_n}x, e^{2\pi i k_n \theta}) : n = 1, 2, \ldots\}$ is dense in $X \times \mathbb{T}$ for every $(r, \theta) \in A$.*

Here $\mathbb{T}$ denotes the unit circle, i.e. $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Proof of Claim. Let $\{x_j : j \in \mathbb{N}\}$, $\{t_l : l \in \mathbb{N}\}$ be dense subsets of X, $\mathbb{T}$ respectively. For every $j, l, s, n \in \mathbb{N}$ define the set

$$A_{j,l,s,n} := \left\{ (r, \theta) \in (1, +\infty) \times \mathbb{R} : \|(rT)^{k_n}x - x_j\| < \frac{1}{s}, |e^{2\pi i k_n \theta} - t_l| < \frac{1}{s} \right\}.$$

We shall prove that the set $A := \bigcap_{j,l,s} \bigcup_n A_{j,l,s,n}$ has the desired properties. Since $A_{j,l,s,n}$ is open we conclude that A is G_δ. Let us show that A is dense in $(1, +\infty) \times \mathbb{R}$. In view of Baire's theorem it suffices to prove that for any fixed $j, l, s \in \mathbb{N}$ the set $\bigcup_n A_{j,l,s,n}$ is dense in $(1, +\infty) \times \mathbb{R}$. To this end, fix $j, l, s \in \mathbb{N}$ and let $b > 1$, $a \in \mathbb{R}$ and $\epsilon > 0$. We seek $r > 0$, $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$ such that

$$|b - r| < \epsilon, \ |a - \theta| < \epsilon, \ |t_l - e^{2\pi i k_n \theta}| \text{ and } \|(rT)^{k_n} x - x_j\| < 1/s.$$

Define the set $B := \{k_n : \|(bT)^{k_n} x - x_j\| < 1/s\}$ and consider its elements in an increasing order, say $k_{\rho_1} < k_{\rho_2} < \cdots$. Of course, we have $B = \{k_{\rho_n} : n \in \mathbb{N}\}$. Now we use Weyl's theorem, see Theorem 4.1 in page 32 from [13], to conclude that the sequence $(k_{\rho_n} \theta)$ is uniformly distributed modulo 1 for almost all θ in $\mathbb{R}$. Hence, there exists $\theta \in \mathbb{R}$ such that the set $\{e^{2\pi i k_{\rho_n} \theta} : n \in \mathbb{N}\}$ is dense in $\mathbb{T}$ and $|a - \theta| < \epsilon$. Finally, setting $r := b$ and from all the above we conclude that there exists $n := \rho_m$ for some $m \in \mathbb{N}$ such that

$$|b - r| < \epsilon, \ |a - \theta| < \epsilon, \ |t_l - e^{2\pi i k_n \theta}| \text{ and } \|(rT)^{k_n} x - x_j\| < 1/s,$$

which is what we wanted to prove. It remains to show that A has full measure in $(1, +\infty) \times \mathbb{R}$. Actually, it is enough to prove that the set $\bigcup_n A_{j,l,s,n}$ has full measure in $(1, +\infty) \times \mathbb{R}$ for every $j, l, s \in \mathbb{N}$. Fix $j, l, s \in \mathbb{N}$ and take any four numbers d_1, d_2, d_3, d_4 with $d_1 < d_2$, $1 < d_3 < d_4$. For any subset B of $(1, +\infty) \times \mathbb{R}$ the symbol B_r stands for its section, i.e. $B_r := \{\theta \in \mathbb{R} : (r, \theta) \in E\}$ and for simplicity reasons we set $E := \bigcup_n A_{j,l,s,n}$. Observe that the proof of denseness result implies that for every $r \in [d_3, d_4]$ we have $(r, \theta) \in E$ for almost every θ in $\mathbb{R}$ (of course the set of such θ's depends on r). It now follows that

$$\mu((E \cap ([d_3, d_4] \times [d_1, d_2]))_r) = d_2 - d_1 = \mu(([d_3, d_4] \times [d_1, d_2])_r) \text{ for every } r \in [d_3, d_4],$$

where μ denotes the Lebesgue measure, and by Cavalieri's principle, see page 149 in [12], we conclude that

$$\mu \times \mu(E \cap ([d_3, d_4] \times [d_1, d_2])) = (d_2 - d_1)(d_4 - d_3).$$

Thus, E has full measure in $(1, +\infty) \times \mathbb{R}$. This completes the proof of the Claim and hence that of Theorem 4.1.

Item (ii) of Theorem 1.1 and Theorem 4.1 directly imply item (iii) of Theorem 1.1.

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